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A THEORY FOR NATURAL CONVECTION TURBULENT BOUNDARY LAYERS NEXT TO HEATED VERTICAL SURFACES

WILLIAM K. GEORGE, JR.† and STEVEN P. CAPP†
Service des Transferts Thermiques, Centre d'Etudes Nucleaires de Grenoble, 38041 Grenoble Cedex, France

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Abstract—The turbulent natural convection boundary layer next to a heated vertical surface is analyzed by classical scaling arguments. It is shown that the fully developed turbulent boundary layer must be treated in two parts: an outer region consisting of most of the boundary layer in which viscous and conduction terms are negligible and an inner region in which the mean convection terms are negligible. The inner layer is identified as a constant heat flux layer.

A similarity analysis yields universal profiles for velocity and temperature in the outer and constant heat flux layers. An asymptotic matching of these profiles in an intermediate layer (the buoyant sublayer) as $H_\delta \equiv g\beta F_0\delta^4/\alpha^3 \to \infty$ yields analytical expressions for the buoyant sublayer profiles as

$$\frac{T-T_w}{\mathbf{T}_l} = K_2\left(\frac{y}{\eta}\right)^{-1/3} + A(Pr),$$

$$\frac{U}{\mathbf{U}_l} = K_1\left(\frac{y}{\eta}\right)^{1/3} + B(Pr),$$

where K_1, K_2 are universal constants and $A(Pr), B(Pr)$ are universal functions of Prandtl number. Asymptotic heat transfer and friction laws are obtained as

$$Nu_x = C'_H(Pr)H_x^{*1/4}, \qquad \tau_w/\rho\mathbf{U}_l^2 = C_f(Pr),$$

where $C'_H(Pr)$ is simply related to $A(Pr)$. Finally, conductive and thermo-viscous sublayers characterized by a linear variation of velocity and temperature are shown to exist at the wall.

All predictions are seen to be in excellent agreement with the abundant experimental data.

NOMENCLATURE

A, universal function (37);
A_1, universal constant (38);
B, universal function (45);
B_1, universal constant (44);
C_f, drag coefficient, $= \dfrac{\tau_w}{\rho U_l^2}$ (54);
C'_H, universal function of Prandtl number, $= Nu_\eta = \dfrac{F_0}{\alpha(T_w - T_\infty)}\left(\dfrac{\alpha^3}{g\beta F_0}\right)^{1/4}$ (42);
C_H, universal function of Prandtl number, $= \left[\dfrac{F_0}{\alpha\Delta T_w}\left(\dfrac{\alpha^2}{g\beta\Delta T_w}\right)^{1/3}\right]$ (61);
C_p, specific heat at constant pressure;
F_0, $\dfrac{q_w}{\rho C_p}$ (7);
g, gravitational acceleration;
H_x^*, H number for constant heat flux wall, $= \left(\dfrac{g\beta F_0 x^4}{\alpha^3}\right)$ (9);
H_x, H number for constant temperature wall, $= \left[\dfrac{g\beta(T_w - T_\infty)x^3}{\alpha^2}\right]$ (62);
K_1, universal constant (26) and (44);
K_2, universal constant (27) and (35);
Nu_l, Nusselt number based on l, $= \left(\dfrac{q_w l}{\alpha\rho C_p\Delta T_w}\right) = \left(\dfrac{F_0 l}{\alpha\Delta T_w}\right)$;
Pr, Prandtl number, $= \nu/\alpha$;
q_w, wall heat flux;
Ra_x^*, modified Rayleigh number, $= \left(\dfrac{g\beta F_0 x^4}{\alpha^2\nu}\right)$ (64);
T, mean temperature;
$\mathbf{T}_l$, inner temperature scale, constant heat flux wall, $= [F_0^{3/4}(gB\alpha)^{-1/4}]$ (25);
$\mathbf{T}_{lT}$, inner temperature scale, constant temperature wall, $= (T_w - T_\infty)$ (58);
$\mathbf{T}_0$, outer temperature scale, $= [F_0^{2/3}(g\beta\delta)^{-1/3}]$ (19);
T_w, wall temperature;
ΔT, $T - T_\infty$;
ΔT_w, $T_w - T_\infty$;
T_∞, reference temperature infinite distance from wall;

†Permanent address: Department of Mechanical Engineering, State University of New York at Buffalo, Buffalo, NY 14214, U.S.A.

U, mean velocity in x direction;
U_M, the maximum velocity;
U_I, inner velocity scale, constant heat flux wall, $= (g\beta F_0 \alpha)^{1/4}$ (24);
U_{IT}, inner velocity scale, constant temperature wall, $= [g\beta(T_w - T_\infty)\alpha]^{1/3}$ (57);
U_0, outer velocity scale, $= (F_0 g\beta\delta)^{1/3}$ (18);
V, mean velocity in y direction;
x, distance in direction parallel to wall;
y, distance in direction perpendicular to wall;
$\bar{y}$, dimensionless outer variable, $= (y/\delta)$ (33);
y^+, dimensionless inner variable, $= (y/\eta)$ (34).

Greek symbols
α, thermal diffusivity;
β, thermal expansion coefficient;
δ, outer length scale;
δ_T, temperature boundary-layer thickness, $= \int_0^\infty \frac{\Delta T}{\Delta T_w} \mathrm{d}y$;
δ_U, velocity boundary-layer thickness, $= \int_0^\infty \frac{U}{U_m} \mathrm{d}y$;
η, inner length scale for constant heat flux wall, $= \left(\frac{\alpha^3}{g\beta F_0}\right)^{1/4}$ (14);
η_T, inner length scale for constant temperature wall, $= \left[\frac{\alpha^2}{g\beta(T_w - T_\infty)}\right]^{1/3}$ (59);
ν, kinematic viscosity;
ρ, density;
τ_w, wall shear stress.

1. INTRODUCTION

THE PROBLEM of the turbulent natural convection boundary-layer flow next to a heated vertical surface has been the subject of numerous investigations. [1]–[18], [28]. In spite of the numerous measurements, there is no consensus on the scaling relations which should be applied to the data or even the basic heat transfer law. The primary reason for this must be the lack of convincing theoretical arguments as to which dimensionless groups govern and which physical phenomena dominate.

All theoretical efforts to data have depended on analogies with the dynamics of the forced flow boundary layer. The earliest attempt to analyze the turbulent natural convection boundary layer on vertical surfaces was due to Eckert and Jackson [10] whose empirical approach depended on an assumed power law temperature profile (later shown to be seriously in error [1]). Bayley [11] carried out a similar analysis which did correctly predict the observed heat-transfer relationship for air, but yielded little insight into the physics of the boundary layer. More recently, there have been a number of partially successful attempts to apply turbulence computational models to the calculation of buoyant flows next to vertical surfaces [12]–[16]. The first three papers used simple eddy viscosity distributions while the last two calculated the eddy viscosity from dynamical equations for the turbulence. The recent work of Raithby [27] is interesting in that in a semi-empirical manner he anticipates the two-layer analysis given here.

Two recent attempts have been made to develop a theory of natural convection boundary layers by proposing scaling laws for the equations of motion. Piau [18] attempted such an analysis using the wall temperature (difference) and the friction velocity for temperature and velocity scales. An outer length scale was chosen to be the boundary-layer thickness while the inner length scale was formed from the friction velocity and the viscosity as in forced flows. While this approach is known to be valid for forced flows (c.f. Monin and Yaglom [19]), it will be seen later to be incorrect here for two reasons: first, the temperature and velocity scales for the inner and outer boundary layer are different and second, the friction velocity is not an independent parameter and is not relevant to the problem. Coutanceau [7] proceeded along different lines but failed to recognize the inner–outer character of the flow; consequently, his scaling laws were valid only for the region closest to the wall.

Finally, we must note the theoretical work of Priestly [20] and others† and the related experiment of Elder [21]. Priestly argued that for natural convection from a vertical flat surface there must exist a region of the flow which is characterized only by the heat flux, the buoyancy parameter and the distance from the surface. It follows immediately on dimensional grounds that the profile of temperature must depend on the inverse cube root of the distance from the wall. Although Priestly's arguments were made for the atmosphere, Elder realized their applicability to at least the natural convection flow between differentially heated vertical parallel plates of high aspect ratio. Although the Rayleigh number of his experiment was so low that the turbulence may not have been fully developed, he did demonstrate the existence of a limited $y^{-1/3}$ range. Unfortunately, this does not appear to have been pursued by other investigators nor have the complete implications for turbulent natural convection next to vertical surfaces been realized until now.

In the remainder of this paper, the fully developed turbulent natural convection boundary layers for constant temperature and constant heat flux vertical surfaces will be analyzed. Outer and inner flow regions will be identified and it will be shown that the inner region is a constant heat flux layer. This constant heat flux layer will be seen to consist of two

† Monin and Yaglom [19] list several other sources for these arguments and state that they were even known to Prandtl.

major subdivisions and a buffer layer between them. Conductive and viscous sublayers in which both velocity and temperature profiles are linear will be shown to exist next to the wall. Another subregion, a buoyant sublayer, will be shown to exist at the outer part of the constant heat flux layer where mean velocity and temperature profiles depend on the cube root and the inverse cube root of distance from the wall respectively. These subregions are illustrated graphically in Fig. 2. Finally, heat transfer and friction laws will be proposed and all predictions will be compared with the abundant experimental data.

PART I: THEORY

2. THE EQUATIONS OF MOTION FOR THE MAIN PART OF THE BOUNDARY LAYER

The flow to be analyzed is shown in Fig. 1. A semi-infinite vertical flat plate is maintained at a spacially uniform and steady heat flux through the plate surface.† For high enough heat flux the flow is known to become unstable at some distance from the leading edge and undergoes transition to a fully developed turbulent flow (c.f. Gebhart [22]). It is this fully developed turbulent state with which we are concerned.

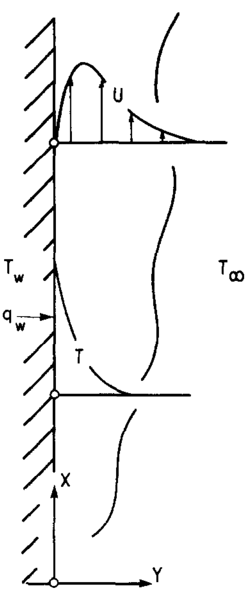

FIG. 1. Schematic of turbulent boundary layer next to heated wall.

We choose the positive x-direction to be opposite to that of the gravitational acceleration and take reference quantities at infinite distance from the plate. Scale quantities for velocity, temperature, and a length scale perpendicular to the plate are defined as $\mathbf{U}_0$, $\mathbf{T}_0$ and δ respectively. By making the usual approximations for boundary-layer flow that $\partial/\partial x \sim 1/L \ll \partial/\partial y \sim 1/\delta$, taking the limit as the Reynolds and Peclet numbers defined by $\mathbf{U}_0\delta/\nu$ and $\mathbf{U}_0\delta/\alpha$

†In Section 11 we shall show that the theory developed below is the same for constant wall temperature flow.

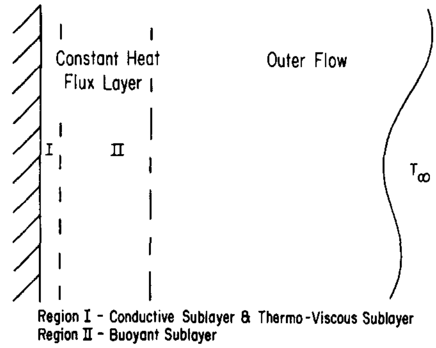

FIG. 2. Diagram showing major regions of the boundary layer. Region I includes conductive and thermo-viscous sublayers. Region II is buoyant sublayer.

become infinite and using the y-momentum equation to eliminate the pressure, the boundary-layer equations of motion to within a Boussinesq approximation can be reduced to

$$U\frac{\partial U}{\partial x}+V\frac{\partial U}{\partial y}\simeq\frac{\partial}{\partial y}[-\overline{uv}]+g\beta(T-T_\infty), \quad (1)$$

$$\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}=0, \quad (2)$$

$$U\frac{\partial T}{\partial x}+V\frac{\partial T}{\partial y}\simeq\frac{\partial}{\partial y}[-\overline{vt}]. \quad (3)$$

Capital letters refer to mean values while lower case letters are used for fluctuating quantities. An overbar is used to denote the ensemble or time average since the flow is assumed statistically stationary.

The largest neglected term in equation (1) is $\partial[\overline{u^2}-\overline{v^2}]/\partial x$, the streamwise gradient of the turbulent normal stress difference. This term is of the order of $\mathbf{u}^2/\mathbf{U}_0^2$ which is small compared to the terms which are retained; $\mathbf{u}$ is a characteristic scale for the velocity fluctuations. In equation (3), the largest neglected term is the streamwise gradient of the turbulent vertical heat flux, $\partial[\overline{ut}]/\partial x$ which is of the order of $\mathbf{ut}/\mathbf{U}_0\mathbf{T}_0$ where $\mathbf{t}$ is a scale characteristic of the turbulent temperature fluctuations. Note that the succeeding analysis is not dependent on the neglect of these terms.

We recognize that these equations can not be valid near the wall because of the absence of the viscous and conductive terms. The problem posed by this absence is particularly serious since the phenomenon being analyzed is a wall phenomenon; that is, all the non-trivial boundary conditions on the solution are imposed at the wall. Therefore, to complete the description of the flow, we must seek a set of equations which retain the viscous and conductive terms and are valid near the wall. Before proceeding to develop these inner equations, note that the outer scales $\mathbf{U}_0$, $\mathbf{T}_0$ and δ have not been specified. To do so, we must first decide which parameters govern the outer flow. This can be accomplished only after the inner or wall problem is solved since all constraints on the flow are imposed at the wall.

3. THE EQUATIONS OF MOTION FOR THE NEAR WALL REGION

Our goal is to rescale the equations of motion so that at least one viscous and one conduction term are retained. If we define new inner velocity and temperature scales as $\mathbf{U}_I$ and $\mathbf{T}_I$, it is straightforward to show that the viscous term can be retained only if the inner length scale is given by $\nu/\mathbf{U}_I$ while retention of the conduction term requires an inner length of scale of $\alpha/\mathbf{U}_I$. We call this new inner length scale η and for now defer the choice between α and ν. By standard scaling procedures (c.f. Tennekes and Lumley [23]), it is straightforward to show that the equations of motion for this inner layer are given by

$$0 \simeq \frac{\partial}{\partial y}\left[-\overline{uv}+\nu\frac{\partial U}{\partial y}\right]+g\beta(T-T_\infty), \tag{4}$$

$$0 \simeq \frac{\partial}{\partial y}\left[-\overline{vt}+\alpha\frac{\partial T}{\partial y}\right]. \tag{5}$$

The pressure term has been eliminated using the y-momentum equation. The leading neglected terms are the mean convection terms which are of order η/L. That this is indeed small will be justified later.

It is appropriate to note that because different inner length scales are needed for the momentum and temperature equations, $\nu/\mathbf{U}_I$ and $\alpha/\mathbf{U}_I$ respectively, certain modifications can be made to equations (4) and (5) if the Prandtl number is large or small compared to unity. For example, for $Pr \ll 1$, there will exist a region close to the wall in which conduction effects are important but a much thinner region in which viscous effects are important. Thus, our inner layer will in reality be two regions: an inner–inner layer where viscous and conduction terms must be included and an outer–inner layer where only the conduction term must be included. Similar arguments can be made for the limit at $Pr \gg 1$. These effects will not affect the arguments to follow as long as no assumptions are made regarding the Prandtl number. Moreover, since equations (4) and (5) contain both the viscous and conduction terms, they will be valid as long as both the Reynolds and Peclet numbers are large.

The momentum equation (4), can be integrated to yield

$$-\overline{uv}+\nu\frac{\partial U}{\partial y}+\int_0^y g\beta(T-T_\infty)\,\mathrm{d}y' = \frac{\tau_w}{\rho}, \tag{6}$$

where τ_w is the wall shear stress. It is clear that because of the presence of the buoyancy integral the inner layer is not a constant stress layer. In forced flow the wall shear stress measures the forcing of the outer flow on the inner and provides the inner boundary condition on the outer flow because of the constant stress layer. In this flow, the wall shear stress is not a fundamental parameter of the flow in the sense that it is imposed on both the inner and outer layers. Therefore it must be treated as a dependent parameter.

Equation (5) for the temperature can similarly be integrated to yield

$$-\overline{vt}+\alpha\frac{\partial T}{\partial y} = -\frac{q_w}{\rho C_p} \equiv -F_0, \tag{7}$$

where q_w is the wall heat flux, C_p is the specific heat at constant pressure, and F_0 is defined by this equation and is at most a function of x. Clearly, the inner layer is a constant heat flux layer in the sense that the total heat flux across the layer is independent of distance from the wall. Thus, the heat flux is a fundamental parameter, not only for the inner layer but also for the outer layer. It is fundamental to the inner layer because it directly measures the "forcing" of the flow by the boundary conditions. Because of the constant heat flux layer it also directly measures the "forcing" of the outer flow by the inner layer. We will use this fact later to determine the outer scales.

4. UNIVERSAL PROFILES FOR VELOCITY AND TEMPERATURE

For the natural convection turbulent boundary layer on a semi-infinite flat plate, the only parameters which can govern the evolution of the flow are those occurring either in the equations of motion or those imposed at the wall. The only parameters occurring in the equations of motion are α, ν and $g\beta$. At any given cross-section, the distance from the leading edge x must be considered as a parameter along with F_0 which is specified at the wall.

From this basic set of five independent parameters, only two independent dimensionless ratios can be formed; we choose

$$Pr = \frac{\nu}{\alpha} \tag{8}$$

and

$$H_x^* = \frac{g\beta F_0 x^4}{\alpha^3}. \tag{9}$$

Equation (8) is readily recognized as the Prandtl number while equation (9) defines what we will call the H-number.

In general, we can write the functional form of the mean velocity and temperature profiles as

$$U = \mathbf{U}_s f_1(y/\delta, H_x, Pr) \tag{10}$$

and

$$T-T_\infty = \mathbf{T}_s f_2(y/\delta, H_x, Pr), \tag{11}$$

where $\mathbf{U}_s$, $\mathbf{T}_s$ and δ are scale quantities chosen from the available parameters.

We hypothesize that the averaged profiles of mean velocity and temperature depend only on the local scales at any given cross-section x; that is, the flow is in local scale equilibrium.† With this, equations (10) and (11) reduce to

$$U = \mathbf{U}_s f_1(y/\delta, \delta/\eta, Pr) \tag{12}$$

†This also applies to all other averaged quantities of interest and can be justified by a similarity scaling of the equations of motion for the inner and outer layers.

and

$$T - T_\infty = \mathbf{T}_s f_2(y/\delta, \delta/\eta, Pr),† \quad (13)$$

where we have defined η to be

$$\eta = \left[\frac{\alpha^3}{g\beta F_0}\right]^{1/4}. \quad (14)$$

If we define an H-number based on the outer length scale δ, it is clear that H_δ is related to the ratio of inner to outer length scales by

$$\frac{\delta}{\eta} = \left[\frac{g\beta F_0 \delta^4}{\alpha^3}\right]^{1/4} = H_\delta^{1/4}. \quad (15)$$

Thus, $\delta/\eta \to \infty$ as $H_\delta \to \infty$ and the H-number plays the role played by the Reynolds number in turbulent forced boundary layers (c.f. Monin and Yaglom [19], Tennekes and Lumley [23]).

In the next few sections we shall be examining the behavior of the profiles given by equations (12) and (13) in the limit as $\delta/\eta \to \infty$. This must be understood to mean that the outer scale δ must be much larger than both the viscous and conduction length scales. Thus, the relevance of the asymptotic analysis based on δ/η to finite H_δ experiments can be expected to be determined in part by the Prandtl number. For example, theory applicable to air flows above a certain value of δ/η might not be applicable to a higher Prandtl number flow until δ/η reaches a considerably larger value.

5. VELOCITY AND TEMPERATURE "DEFICIT" LAWS

We look first at the outer region where $y \sim \delta$. We have previously shown that the equations of motion in this region are independent of the Prandtl number. Also, for the velocity and temperature profiles to be well-behaved as $\delta/\eta \to \infty$, the functions f_1 and f_2 must be asymptotically independent of δ/η. Therefore, the functional forms reduce to

$$U - U_M = \mathbf{U}_0 f_{10}(y/\delta), \quad (16)$$

$$T - T_\infty = \mathbf{T}_0 f_{20}(y/\delta), \quad (17)$$

where the subscript 0 is used to emphasize the outer character of these functions. The velocity must be referenced to a non-zero outer layer velocity to avoid the necessity of accounting for the velocity change across the inner layer thereby introducing a dependence on α and ν. In forced convection we accomplish this by referencing to U_∞; here we use U_M, the velocity maximum.

The scale quantities $\mathbf{U}_0$ and $\mathbf{T}_0$ must be entirely determined by local parameters. The only parameters which are relevant in this region are the buoyancy $g\beta$ and the heat flux F_0 which is imposed at the wall and is unchanged by the inner layer. On dimensional grounds, the only choices for $\mathbf{U}_0$ and $\mathbf{T}_0$ are

$$U_0 = (F_0 g\beta\delta)^{1/3} \quad (18)$$

†As long as the Prandtl number dependence is retained, the problems presented by this choice at high or low Prandtl number are avoided.

and

$$\mathbf{T}_0 = F_0^{2/3}(g\beta\delta)^{-1/3}, \quad (19)$$

where δ is the local length scale which can be x-dependent.

Equations (16–19) define the velocity and temperature "deficit" laws for natural convection boundary layers next to heated vertical surfaces. The analogy with forced convection flows is clear (c.f. Monin and Yaglom [19]).

6. A "LAW OF THE WALL"

We consider now the inner layer where $y \ll \delta$. We note first that the Prandtl number dependence must be retained in the functional forms since it occurs explicitly in the non-dimensionalized equations for the inner layer. We can reformulate the functional forms of equations (12) and (13) to better reflect the inner character of this region. Equivalently, we can write

$$U = \mathbf{U}_I f_{1I}(y/\eta, \delta/\eta, Pr) \quad (20)$$

and

$$T - T_w = \mathbf{T}_I f_{2I}(y/\eta, \delta/\eta, Pr), \quad (21)$$

where subscript I is used to emphasize the inner character of these functions and the temperature is referenced to the wall temperature T_w.

If $y \sim \eta$ while $\delta/\eta \to \infty$, the functional dependence on δ/η must vanish if the functions are to remain well-behaved. Thus, we can write for the inner layer

$$U = \mathbf{U}_I f_{1I}(y/\eta, Pr) \quad (22)$$

and

$$T - T_w = \mathbf{T}_I f_{2I}(y/\eta, Pr). \quad (23)$$

The reason for referencing the temperature to the wall temperature is now clear since equation (23) depends only on inner quantities and does not depend on the part of temperature drop across the outer layer.

The scale quantities $\mathbf{U}_I$ and $\mathbf{T}_I$ can depend only on ν, α, $g\beta$ and F_0 since these are the only available independent parameters. On dimensional grounds

$$\mathbf{U}_I = (g\beta F_0 \alpha)^{1/4} \quad (24)$$

and

$$\mathbf{T}_I = F_0^{3/4}(g\beta\alpha)^{-1/4}, \quad (25)$$

where we have used α instead of ν for experimental convenience.† Since equations (22) and (23) retain a Prandtl number dependence the choice is immaterial.

Equations (22–25) formulate a statement of the law of the wall for natural convection boundary layers next to vertical surfaces. The region of applicability of this law corresponds exactly to the constant heat flux layer obtained earlier. The analogy with forced flow boundary layers is clear (c.f.

†In air $\nu \sim \alpha$ whereas in liquids ν tends to be more strongly temperature dependent than α.

[19]). It is important to note that the Prandtl number occurs explicitly in the equations for the inner layer whereas it does not occur in the outer layer equations because of the high Reynolds and Peclet number assumptions.

7. THE BUOYANT SUBLAYER

Suppose there exists a flow region which is sufficiently far from the wall that the viscous and conduction effects are negligible but yet close enough to the wall that the mean convection effects are not important. In effect we require $\eta \ll y \ll \delta$ and $Pr^{3/4} \cdot \eta \ll y \ll \delta$. If such a region exists, the only governing parameters can be F_0 and $g\beta$. Thus, gradients of mean velocity and temperature can depend only on $g\beta$, F_0 and the distance to the wall. Dimensionally, we must have

$$\frac{dU}{dy} = \frac{K_1}{3} \cdot \left[\frac{g\beta F_0}{y^2}\right]^{1/3} \tag{26}$$

and

$$\frac{dT}{dy} = -\frac{K_2}{3}\left[\frac{F_0^2}{g\beta y^4}\right]^{1/3} \tag{27}$$

where K_1 and K_2 are absolute constants and the numerical factors are chosen for convenience.

These equations can be integrated directly to yield

$$U = K_1(g\beta F_0 y)^{1/3} + B'' \tag{28}$$

and

$$T - T_\infty = K_2\left[\frac{F_0^2}{g\beta}\right]^{1/3} y^{-1/3} + A''. \tag{29}$$

The integration constants A'' and B'' must be functions of the thermal diffusivity and the kinematic viscosity since these account for the effect of the wall layer.

We call this intermediate layer the buoyant sublayer by direct analogy with the inertial sublayer of forced flows. It is clear that its existence is entirely dependent on the magnitude of H_δ (or δ/η). The buoyant sublayer can be identified as the outer part of the constant heat flux layer in exactly the same manner as the inertial (or logarithmic) sublayer is identified with the outer part of the constant stress layer in forced flows.

8. ALTERNATE APPROACH TO THE BUOYANT SUBLAYER

In this section we will derive the buoyant sublayer profile in a somewhat more formal manner than the purely heuristic approach above. We begin by asking whether there is a matched layer in which at fixed y as the length scale ratio $\delta/\eta \to \infty$, the inner limit of the outer solution is equal to the outer limit of the inner solution. This approach has been used to derive the logarithmic profile for the inertial sublayer of forced flows (c.f. Tennekes and Lumley [23]). Physically, we are asking whether there is a flow region in which only the turbulent heat transfer and buoyancy are important.

For the inner layer near the wall, we have for the mean temperature

$$\frac{T - T_w}{\mathbf{T}_i} = f_{2i}(y/\eta, Pr) \tag{30}$$

and for the outer layer

$$\frac{T - T_\infty}{\mathbf{T}_0} = f_{2o}(y/\delta). \tag{31}$$

Restricting ourselves (for now) to the case where the wall heat flux is specified, differentiating (30) and (31) and equating derivatives, it can be shown that the matching condition is

$$y^{+4/3} f'_{2i}(y^+, Pr) = \tilde{y}^{4/3} f'_{2o}(\tilde{y}), \tag{32}$$

where we have defined

$$\tilde{y} = y/\delta \tag{33}$$

and

$$y^+ = y/\eta. \tag{34}$$

It is immediately obvious that the two sides of equation (32) are functions of independent variables in the limit as $\delta/\eta \to \infty$. It follows immediately that in this limit, both sides of the equation must equal an absolute constant, say $-K_2/3$; that is,

$$\frac{df_{2i}}{dy^+} = -\frac{1}{3}\frac{K_2}{(y^+)^{4/3}} \tag{35}$$

and

$$\frac{df_{2o}}{d\tilde{y}} = -\frac{1}{3}\frac{K_2}{\tilde{y}^{4/3}}. \tag{36}$$

These can be integrated directly to yield

$$f_{2i} = K_2(y^+)^{-1/3} + A(Pr) \tag{37}$$

and

$$f_{2o} = K_2(\tilde{y})^{-1/3} + A_1 \tag{38}$$

or in physical variables

$$\frac{(T - T_w)(g\beta\alpha)^{1/4}}{F_0^{3/4}} = K_2(y/\eta)^{-1/3} + A(Pr) \tag{39}$$

and

$$\frac{(T - T_\infty)(g\beta\delta)^{1/3}}{F_0^{2/3}} = K_2(y/\delta)^{-1/3} + A_1. \tag{40}$$

K_2 and A_1 are universal constants and A is a universal function of the Prandtl number.

Hence, there exists a matched layer and in it the mean temperature profile varies as the inverse cube root of the distance from the wall. The constants A_1, K_2 and the function $A(Pr)$ can be determined from experiment. It is clear that this information is equivalent to the result derived in equation (29); however, by imposing the physical constraint that both the inner and outer representations yield the same temperature at a given point in the matched layer we obtain a considerable bonus.

This condition can be satisfied only if

$$C_H'^{-1} = A_1\left[\frac{\eta}{\delta}\right]^{1/3} - A(Pr), \qquad (41)$$

where C_H' is defined by

$$C_H' = \frac{\mathbf{T}_I}{T_w - T_\infty} = \frac{F_0}{\alpha(T_w - T_\infty)}\left(\frac{\alpha^3}{g\beta F_0}\right)^{1/4} = Nu_\eta. \qquad (42)$$

Nu_η is readily recognized as the Nusselt number based, in this case, on the length scale η. It is clear that as $H_\delta = \delta/\eta \to \infty$, $(\eta/\delta)^{1/3} \to 0$, and $C_H'^{-1} \to -A(Pr)$. Thus C_H' is a function of Prandtl number only.

It follows immediately from equation (42) that the heat-transfer coefficient is independent of δ (and hence x) in the limit as $H_\delta \to \infty$. Thus the heat-transfer law (expressed in the usual manner) is asymptotically given by

$$Nu_x = C_H'(Pr)\cdot H_x^{*1/4}. \qquad (43)$$

Thus the asymptotic matching of inner and outer temperature profiles has yielded not only the temperature profiles in the buoyant sublayer (and thereby confirmed its existence) but also the heat-transfer law.† The logarithmic heat-transfer law for forced flow follows from similar considerations (c.f. [19]).

A similar matching exercise for the velocity derivatives can be carried out to obtain

$$f_{1O}(\bar{y}) = K_1\bar{y}^{1/3} + B_1, \qquad (44)$$

$$f_{1I}(y^+) = K_1 y^{+1/3} + B(Pr) \qquad (45)$$

or in physical variables

$$\frac{U - U_M}{(g\beta F_0\delta)^{1/3}} = K_1(y/\delta)^{1/3} + B_1 \qquad (46)$$

$$\frac{U}{(g\beta F_0\alpha)^{1/4}} = K_1(y/\eta)^{1/3} + B(Pr). \qquad (47)$$

Requiring that the profiles themselves match yields the following constraint

$$\frac{U_M}{\mathbf{U}_0} = B(Pr)\left(\frac{\eta}{\delta}\right)^{1/3} - B_1. \qquad (48)$$

It is clear that as $\delta/\eta \to \infty$, $U_M/\mathbf{U}_0 \to -B_1$. This result could have been expected in this limit since the maximum velocity occurs in the outer flow and should therefore be proportional to $\mathbf{U}_0$.

Before concluding this section, it is worth noting that the relationships derived above are critically dependent on the choice of inner and outer scales. For example, if the inner and outer scales are chosen to be the same as in [18], logarithmic profiles result, regardless of the particular choices. The choices made here have been seen to be dictated directly by the dynamics of the problem.

†An interesting consequence of equation (41) is that we have obtained the first correction term for finite H_δ. A similar result in equation (47) for the velocity maximum also shows the residual effect of low H_δ.

9. THE WALL HEAT FLUX AND THE TEMPERATURE PROFILE NEAR THE WALL

In the constant heat flux layer the mean temperature equation has been seen to be given by equation (7). As $y \to 0$, the kinematic boundary condition on the velocity at the wall requires that $\overline{vt} \to 0$. Therefore, there must be a region adjacent to the wall in which conduction dominates. Thus for y near the wall, we have a conductive sublayer in which

$$\alpha\frac{\partial T}{\partial y} \cong -\frac{q_w}{\rho C_p} = -F_0. \qquad (49)$$

Integrating from the wall we obtain the dimensionless form

$$\frac{T - T_\infty}{T_w - T_\infty} = 1 - C_H'\cdot\left(\frac{y}{\eta}\right). \qquad (50)$$

Alternately, non-dimensionalizing by the previously defined inner temperature scale $\mathbf{T}_I$ we obtain

$$\frac{T - T_w}{\mathbf{T}_I} = -\frac{y}{\eta}. \qquad (51)$$

Both of these forms will be seen to be useful for comparison with experimental data. Clearly, the temperature profile is linear at the wall. A similar conclusion was reached by Nee and Yang [15] and Coutanceau [7]. Note that the extent of this linear region will clearly depend on the Prandtl number.

10. THE VELOCITY PROFILE NEAR THE WALL

We saw that the momentum equation in the constant heat flux layer could be written as equation (6). We now confine our attention to the region near the wall where because of the kinetic and no-slip boundary conditions $\overline{uv} \to 0$. To emphasize the thermally driven nature of this layer and to distinguish it from its forced flow counterpart, we will call this layer the thermo-viscous sublayer.

Using the no-slip condition at the wall, neglecting the Reynolds stress term and using equation (50) for the temperature, we obtain

$$U = \left(\frac{\tau_w}{\mu}\right)y - \frac{1}{2}\left(\frac{g\beta(T_w - T_\infty)}{\nu}\right)y^2 + \frac{1}{6}\left[\frac{g\beta(T_w - T_\infty)C_H'}{\nu\eta}\right]y^3. \qquad (52)$$

Clearly, the leading term is linear contrary to the statement of [4]. This equation is exactly that obtained by Nee and Yang [15]. Note that the region of validity of this law is limited by two considerations: first, the extent of the linear region in the temperature profile and second, the extent of the region in which the Reynolds stresses are negligible. Both of these will depend on the Prandtl number.

We can write equation (52) for the velocity at the wall in dimensionless form as

$$\frac{U}{\mathbf{U}_I} = Pr^{-1}\left\{C_f y^+ - \frac{C_H'^{-1}}{2}y^{+2} + \tfrac{1}{6}y^{+3}\right\}, \qquad (53)$$

where we have defined a friction coefficient C_f as

$$C_f \equiv \frac{\tau_w}{\rho \mathbf{U}_I^2} = \frac{u_*^2}{\mathbf{U}_I^2}, \tag{54}$$

where u_* is the friction velocity.

Since the velocity profile in the inner layer is asymptotically independent of the outer flow (and hence x), equation (53) must also be independent of the outer flow. In particular, the velocity gradient at the wall, the wall shear stress and hence C_f must be asymptotically functions of the Prandtl number only. Thus it is clear that our inner velocity scale $\mathbf{U}_I$ is to within a function of Prandtl number proportional to the friction velocity and we could have used u_* for the inner velocity scale (but not the outer!). There are three reasons why this would be an inconvenient choice: first, it is not a primitive variable as is $\mathbf{U}_I$ since it is not easily determined from the specified boundary conditions; second, the unknown function of Prandtl number $C_f(Pr)$ would make the matching of Section 8 more difficult and third, in experiments of this type one can seldom (if ever) measure the velocity gradient at the wall accurately enough to use the wall shear stress as a scaling parameter for data.

11. THE ASYMPTOTIC EQUIVALENCE OF CONSTANT WALL TEMPERATURE AND CONSTANT WALL HEAT FLUX FLOWS

For a number of years there has been a suspicion that buoyancy-induced flows at constant wall temperature and constant wall heat flux were closely related (c.f. Vliet and Liu [3]). From equations (41) and (42) which were derived for constant wall heat flux boundary layers it followed that the heat-transfer coefficient defined by $h = q_w/(T_w - T_\infty)$ was independent of x in the limit as $H_x^* \to \infty$. Clearly this can be true only if both q_w and $T_w - T_\infty$ are constant. Thus, in this limit (corresponding to large x) the constant wall heat flux and constant wall temperature boundary layers must be identical.

Since a number of experiments have been performed at constant wall temperature it is convenient to use inner forms for the law of the wall which contain $(T_w - T_\infty)$ instead of F_0. These are given by

$$U = \mathbf{U}_{IT} f_{1IT}(y/\eta_T, Pr), \tag{55}$$

$$T - T_w = \mathbf{T}_{IT} f_{2IT}(y/\eta_T, Pr), \tag{56}$$

where

$$\mathbf{U}_{IT} = [g\beta(T_w - T_\infty)\alpha]^{1/3}, \tag{57}$$

$$\mathbf{T}_{IT} = (T_w - T_\infty) \tag{58}$$

and

$$\eta_T = \left[\frac{\alpha^2}{g\beta(T_w - T_\infty)}\right]^{1/3} \tag{59}$$

A disadvantage of scaling with $T_w - T_\infty$ is that the slope of the profiles in the buoyant sublayer are now Prandtl number dependent. This results from the Prandtl number dependence of C_H'.

It is straightforward to show that the heat-transfer law of equation (43) can be expressed in constant wall temperature variables as

$$Nu_x = C_H(Pr) H_x^{1/3}, \tag{60}$$

where

$$C_H(Pr) = \left[\frac{F_0}{\alpha \Delta T_w}\left(\frac{\alpha^2}{g\beta \Delta T_w}\right)^{1/3}\right] = [C_H']^{4/3} \tag{61}$$

and

$$H_x = \frac{g\beta(T_w - T_\infty)x^3}{\alpha^2}. \tag{62}$$

12. THE UNIVERSALITY OF THE CONSTANT HEAT FLUX LAYER

It is easy to show that equations (4) and (5) will describe the inner layer even when the wall heat flux (or temperature difference) is not constant as long as the changes in the x-direction are more gradual than those in the y-direction. It immediately follows that the profiles derived for the constant heat flux layer have a universal applicability to all turbulent natural convection flows next to vertical surfaces, regardless of the particulars of the boundary conditions. Thus we can expect to find the buoyant sublayer profiles for temperature and velocity, the conductive and thermo-viscous sublayer profiles and even the local heat transfer and friction laws applicable to a wide variety of turbulent flows next to vertical walls. This fact again has its counterpart in forced flows where the logarithmic profiles find universal applicability to a variety of internal and external flow geometries.

PART II: COMPARISON WITH EXPERIMENT

13. THE HEAT-TRANSFER LAW

Fujii *et al.* [5] conducted an extensive series of experiments using vertical cylinders at constant wall temperature. Since the heat transfer at the wall is entirely governed by the wall layer, we can expect that the heat-transfer relation will be the same as that for plates as long as the radius of curvature is much greater than the wall layer thickness.

Figure (3) is replotted from Fig. 14 of reference [5] and includes measurements in water, spindle oil, Mobil-therm oil and ethylene glycol, in addition to Cheesewright's air data. The Prandtl number range for these measurements is 0.7–180. To account for

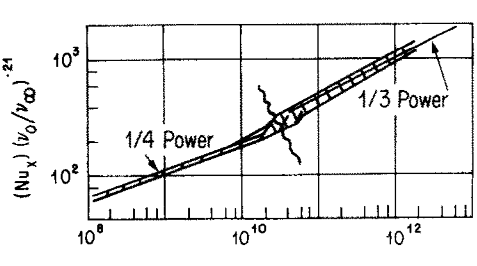

FIG. 3. Heated transfer law in equation (64) is solid line with slope of +1/3. Shaded area shows data (adapted from reference [5]).

the variation of fluid properties over the boundary layer, Fujii and his co-workers suggested using the temperature loading factor $(\nu_{wall}/\nu_\infty)^{0.21}$ and evaluating all other properties at infinity. The turbulent data in Fig. (3) collapse remarkably around the law

$$(Nu_x)_\infty\left(\frac{\nu_w}{\nu_\infty}\right)^{0.21} = 0.13(Pr^{-1/3}H_x^{1/3})_\infty = 0.13(Ra_x)_\infty^{1/3}, \qquad (63)$$

where the subscripts w and ∞ mean that the fluid properties are evaluated at the wall and at infinity, respectively. This is consistent with the prediction of equation (60).

Before considering the measurements at constant wall heat flux, let us first consider the difference between constant wall heat flux and constant wall temperature flows undergoing transition to turbulence. For a constant wall temperature flow, the increased lateral heat transport that accompanies transition can be accommodated by a large increase in the wall heat flux. Thus, the fully developed turbulent flow (and, in particular, the inner layer) can very rapidly be established. This is not true, however, for the constant wall heat flux flow and as a consequence, the temperature must drop catastrophically. It is clear from the measurements of Vliet and Liu [3] and Fujii *et al.* [5] that the wall temperature drops below its equilibrium value and slowly recovers. Thus, although both flows require a development distance to approach equilibrium, the constant wall heat flux case requires longer.

It is easy to show that the effect of allowing measurements in this developing region to influence the choice of an exponent for a heat-transfer law will always be an exponent which is too low. Such was the case in the measurements of Vliet and Liu [3] who obtained $n \sim 0.22$–0.24 for flow in water. It seems safe to conclude that the proposed law (equations 42, 43) is valid when the flow is sufficiently developed.

Since no attempt was made by Vliet and Liu to account for viscosity variation across the flow, a direct comparison is not possible with the equivalent form of equation (63) which is

$$(Nu_x)_\infty\left(\frac{\nu_w}{\nu_\infty}\right)^{0.16} = 0.22(Ra_x^*)_\infty^{1/4}. \qquad (64)$$

It might be significant, however, that the coefficient (0.22) is close to Vliet and Liu's cold water data correlation.

In summary, we can state that there is abundant evidence that the proposed heat-transfer law is valid. Moreover, at least for Prandtl numbers greater than 0.7, it appears that the unknown function of Prandtl number in equation (60) is approximately given by

$$C_H(Pr) = 0.13Pr^{-1/3}; \quad Pr > 0.7. \qquad (65)$$

The coefficient may not correspond to the true asymptotic value in view of the relatively low Rayleigh numbers of the experiments.

14. THE LAW OF THE WALL

The most striking confirmation existing in the literature of the universality of the temperature profile near the wall is due to Fujii *et al.* [5] who plotted $\Delta T/\Delta T_w$ vs $F_0 y/\alpha\Delta T_w$ for their cylinder data. It follows immediately from equation (42) that $F_0/\alpha\Delta T_w \propto \eta_T$. Thus, to within a function of Prandtl number, the plot is precisely that suggested by equation (56). Fujii and his co-workers found that for a narrow range of Prandtl numbers, the profiles measured over a wide range of wall conditions collapsed onto a single curve over most of the boundary layer. The profiles for substantially different Prandtl numbers were different, however, as expected.

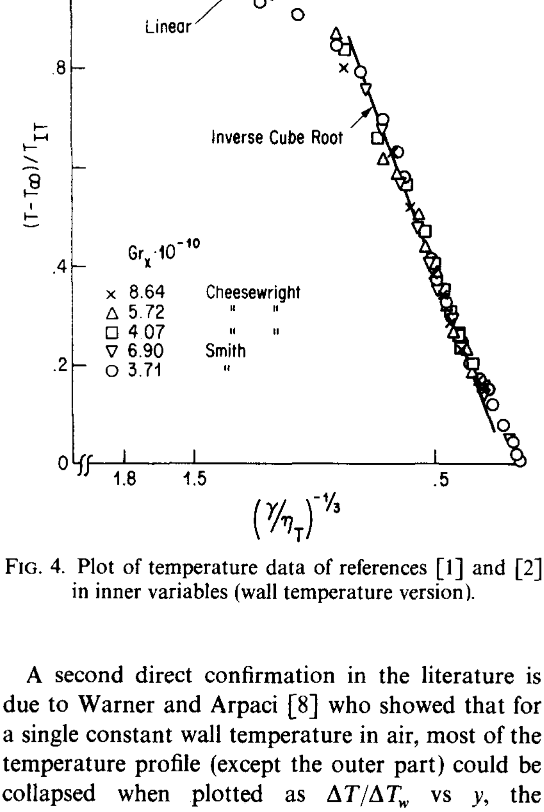

FIG. 4. Plot of temperature data of references [1] and [2] in inner variables (wall temperature version).

A second direct confirmation in the literature is due to Warner and Arpaci [8] who showed that for a single constant wall temperature in air, most of the temperature profile (except the outer part) could be collapsed when plotted as $\Delta T/\Delta T_w$ vs y, the dimensional distance from the wall. Since α, ν, $g\beta$ and ΔT_w were fixed, this is what would have been expected from the arguments presented herein.

Figure 4 shows replots in inner variables of the temperature profiles measured by Cheesewright [1] and Smith [2] in air next to a constant temperature wall. In Cheesewright's experiment, both distance along the plate and wall temperature difference were varied while Smith varied only the distance. The temperature profile is plotted as $\Delta T/\Delta T_w$ vs $(y/\eta_T)^{-1/3}$ so that the buoyant subrange appears as a straight line. The data not only collapse to a single curve, but also exhibit a well-defined linear region next to the wall and an inverse cube root region where we would expect the buoyant sublayer to appear. The temperature profile to an excellent approximation is given by

$$\frac{\Delta T}{\Delta T_w} = \begin{cases} 1-0.1(y/\eta_T) & ;\ 0 \leqslant y/\eta_T \leqslant 1.7 \\ 1.45(y/\eta_T)^{-1/3} - 0.35; & 1.7 < y/\eta_T < D, \end{cases} \quad (66)$$

where D increases as H_x increases.† The inner breakpoint at $y/\eta_T \simeq 1.7$ should not be expected for different Prandtl numbers. From the considerations following equation (42) there is about a 30% discrepancy between the $A(Pr)$ deduced from this expression and the value computed from the $C_H(Pr)$ deduced in equation (65); this is most likely due to the fact that the fully developed state is not reached in the experiments.

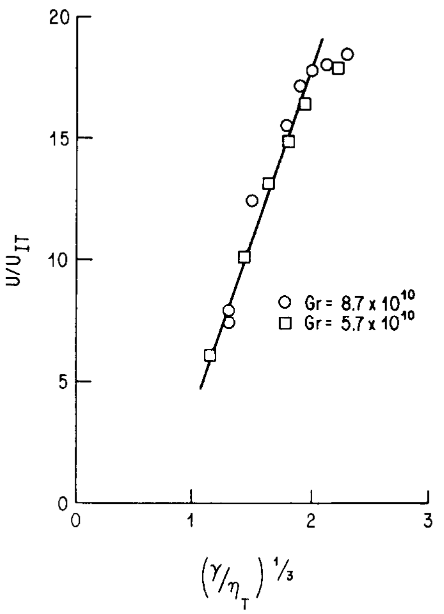

FIG. 5. Plot of velocity data from [2] in inner variables (wall temperature version).

The velocity data of Smith [2] is plotted in Fig. 5 as $U/(g\beta\Delta T_w\alpha)^{1/3}$ vs $(y/\eta_T)^{1/3}$ so that again the buoyant subrange will appear as a straight line. The velocity data (not shown) of Cheesewright exhibit the same internal consistency as that due to Smith but have about a 10% greater slope in the buoyant sublayer. In view of the great difficulty in measuring in heated flow at low velocities and close to surfaces and in view of the obvious internal consistency of both sets of data, we attribute this to a difference in calibration.

It is clear from the velocity plots that there is a well-defined cube root region extending from about $y/\eta_T \simeq 1.7$ which corresponds to that of the temperature profile. The best estimate for the form of the velocity profile in the buoyant subrange is

$$U/(g\beta\Delta T_w\alpha)^{1/3} = 12.3(y/\eta_T)^{1/3} - 9.3; \quad 1.7 \leqslant y/\eta_T < D. \quad (67)$$

†Since the outer profile must scale in outer variables always, it is easy to show that $D \cdot \eta_T/\delta$ const.

Note that in these variables the slope of 12.3 is valid for $Pr = 0.7$ only. Also, it is clear from the velocity plots that there are no points close enough to the wall to expect a linear range or to estimate the friction coefficient.

The point of departure of the velocity profile from a cube root dependence as y increases is somewhat earlier than for the temperature profile. This is not surprising since a boundary-layer flow is always developing and since the development in this case is driven by the temperature. This interpretation is consistent with the fact that the extent of the buoyant subrange for the velocity profile increases with x when plotted in inner variables. Also, it should be noted that the lowest Grashof number profiles reported by both Cheesewright and Smith were not plotted since the inner layer clearly was not fully developed.

Figures 6 and 7 show plots of the temperature and velocity in the heat flux version of the inner variables. In these plots of $\Delta T_w(g\beta\alpha)^{1/4}/F_0^{3/4}$ vs $(y/\eta)^{-1/3}$ and $U/(g\beta F_0\alpha)^{1/4}$ vs $(y/\eta)^{1/3}$, the slope of the buoyant subrange should be Prandtl number independent. In addition to the previously cited data we have also plotted temperature data in water (Fujii *et al.* [5]) and the velocity data in water (Vliet and Liu [3] and mercury (Welty and Peinecke [17]). Because of the scatter in the data, it is not possible to make a reasonable determination of the Prandtl number dependence of the $A(Pr)$ and $B(Pr)$ occurring in equations (39) and (47). It does appear that these functions increase monotonically with Prandtl number. The slopes determined from the air data are given by $K_1 \simeq 27$ and $K_2 \simeq 5.6$. Recall that these should be universal constants. Recent experiments by Qureshi and Gebhart [28] in water confirm this value for K_2.

15. THE DEFICIT LAWS

The most extensive attempt to date to collapse the velocity and temperature profiles to a single curve is due to Vliet and Liu [3]. They plotted the data of several investigations (including their own) as $\Delta T/\Delta T_w$ vs y/δ_T and U/U_M vs y/δ_u where U_M is the maximum velocity and δ_T, δ_u are boundary-layer thicknesses defined by integrating the temperature and velocity profiles. It is straightforward to show by splitting the integrals and using the previously defined inner and outer universal profiles that δ_u is in fact a legitimate measure of the outer length scale δ. δ_T, on the other hand, can be shown to be neither an inner nor outer length scale since most of the temperature drop occurs in the buoyant sublayer. As a consequence, it should at most collapse the data only over a limited intermediate range of distances from the wall. This is consistent with the observations of Vliet and Liu [3]. Since δ_T has no dynamical significance, its use is not recommended.

Figure 8 is adapted from the paper by Vliet and Liu [3] and shows velocity profiles measured in air and water. The authors point out the excellent

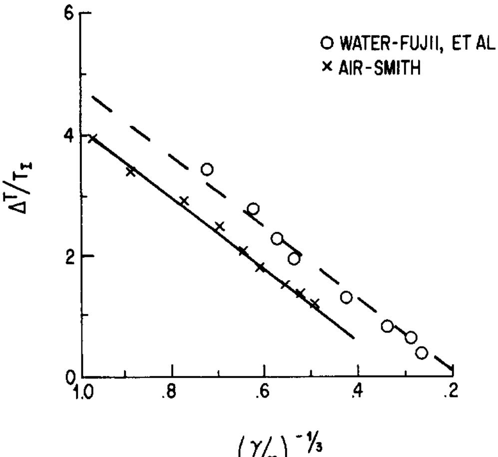

FIG. 6. Plot of temperature data from [1, 2, 5] in inner variables (heat flux version).

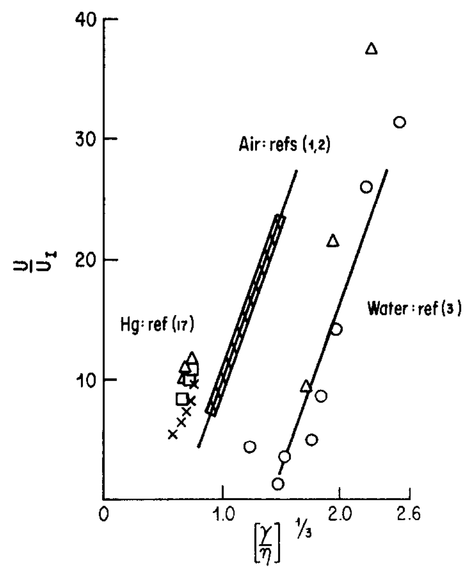

FIG. 7. Plot of velocity data from [2, 3, 17] in inner variables (heat flux version).

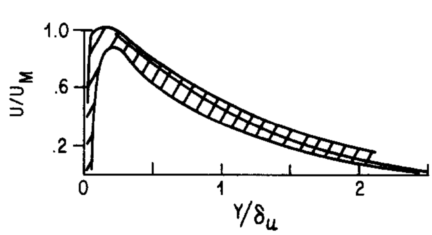

FIG. 8. Plot of velocity from [1, 3] in outer variables (adapted from [3]).

collapse of the data over the outer 90% of the boundary layer. Table 1 summarizes the parameters used in the scaling and calculates the ratios U_M/U_0. The data are in excellent agreement with the suggested outer scaling. The discrepancy of the U_M/U_0 ratios is not surprising in view of the possible integration errors, a lack of knowledge about the actual parameters used by Vliet and Liu and by Cheesewright and perhaps more importantly, the relatively low Rayleigh numbers of the experiments.

Table 1.

	δ_u (cm)	U_M (cm/s)	U_0 (cm/s)	U_M/U_0
Air	7.16	87.8	10.1	8.73
Water	3.81	4.60	0.547	8.41
Water	2.08	3.66	0.447	8.19

It does not seem possible to construct an outer scale plot for the temperature from the data in the literature because of the inaccuracies in measurement at large distances from the wall and a lack of information about the basic parameters in the respective experiments. The trends in the measurements appear to be consistent with the predictions made here, however.

In summary, although the data is sparse, the proposed outer scaling and the deficit laws appear to be valid. It should be possible to construct composite outer flows by combining the buoyant sublayer profiles and empirical fits to the outer flow in the manner which the well-known wake functions for forced flow have been constructed (c.f. Monin and Yaglom [19]).

16. SUMMARY AND CONCLUSIONS

Proceeding from the averaged equations of motion, we have seen the necessity of treating the turbulent natural convection boundary layer on a vertical surface in two parts—an outer flow which is independent of conduction and viscosity effects (equations 1–3), and an inner flow in which the mean convection of momentum and heat is negligible (equations 4 and 5). We have seen that this inner layer is distinguished by having a constant total heat flux; hence, its name, the constant heat flux layer.

The local scale equilibrium of the boundary layer allowed the derivation of universal profiles for velocity and temperature for the inner and outer regions. An overlap region of common validity was shown to exist at high values of H_δ and was termed the buoyant sublayer. In this region, the velocity and temperature were dependent on the cube root and inverse cube root of distance from the wall, respectively. Regions of linear variation of velocity and temperature were seen to exist next to the wall and were termed the conductive and thermo-viscous sublayers, respectively. Finally, heat transfer and friction laws were derived for the fully developed boundary layer.

An attempt was made to compare predictions with the abundant experimental evidence. It was seen that many of the conclusions of this paper could be substantiated directly from the literature and others followed immediately from replots of the data. In particular, the heat-transfer law, the conductive sublayer, and the existence of the buoyant sublayer can be accepted as fact. In addition, agreement with the outer scaling laws was shown to be highly probable from the data. Because of the scatter in the data, the unavailability of needed parameters and the relatively low Rayleigh numbers at which the experiments were performed, firm quantitative conclusions about most of the constants were not possible.

In conclusion, perhaps the most significant contribution of this paper has been that it establishes a framework within which definitive experiments can be made. We have seen here that many of the predictions made here existed in fragmentary form in the literature. Now that these fragments have been united into a single coherent whole, a new generation of experiments can be carried out which are designed specifically to fill in the missing information. Additional theoretical work could be carried out to calculate the functions $A(Pr)$, $B(Pr)$ and $C_f(Pr)$, perhaps from semi-empirical theories of turbulence. This is particularly important since the confirmed existence of a buoyant sublayer can give rise to new computational models for the outer flow which utilize the cube root profiles as inner boundary conditions, thereby avoiding the difficult problem of modeling the turbulence in the inner layer. The extension of the developments here to include rough walls is straightforward and is outlined in [29].

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UNE THEORIE DE LA COUCHE LIMITE TURBULENTE DE CONVECTION NATURELLE PRES DES SURFACES VERTICALES ET CHAUDES

Résumé—La couche limite turbulente de convection naturelle près d'une surface verticale et chaude est analysée par des arguments classiques. On montre que la couche limite turbulente établie peut être traitée en deux parties: une région externe représentant la plupart de la couche limite et dans laquelle les termes de viscosité et de conduction sont négligeables, et une région interne dans laquelle les termes de convection moyenne sont négligeables. La couche intérieure est identifiée à une couche avec flux thermique constant.

Une analyse semblable donne des profils universels pour la vitesse et la température dans les deux couches. Une étude asymptotique de ces profils dans la couche intermédiaire, lorsque $H_\delta = g\beta F_0\delta^4/\alpha^3 \to \infty$, donne des expressions analytiques pour les profils de cette couche:

$$\frac{T-T_w}{T_l} = K_2\left(\frac{y}{\eta}\right)^{-1/3} + A(Pr)$$

$$\frac{U}{U_l} = K_1\left(\frac{y}{\eta}\right)^{1/3} + B(Pr),$$

où K_1, K_2 sont des constantes universelles et $A(Pr)$ et $B(Pr)$ sont des fonctions universelles du nombre de Prandtl. Des lois asymptotiques de transfert thermique et de frottement sont obtenues:

$$Nu_x = C'_H(Pr)H_x^{*1/4}, \quad \tau_w/\rho U_l^2 = C_f(Pr),$$

où $C'_H(Pr)$ est simplement relié à $A(Pr)$. Enfin on montre qu'il existe, à la paroi, des sous-couches conductives et thermo-visqueuses caractérisées par une variation linéaire de la vitesse et de la température.

Toutes les estimations sont en excellent accord avec les données expérimentales abondantes.

EINE THEORIE FÜR TURBULENTE GRENZSCHICHTEN BEI FREIER KONVEKTION AN BEHEIZTEN SENKRECHTEN FLÄCHEN

Zusammenfassung—Die turbulente natürliche Konvektionsgrenzschicht, die an eine beheizte vertikale Fläche angrenzt, wird mit der klassischen Ähnlichkeitstheorie untersucht. Es wird gezeigt, daß die vollständig ausgebildete turbulente Grenzschicht in zwei Bereichen behandelt werden muß: in einem äuß eren Bereich, der den größten Teil der Grenzschicht ausmacht und wo die Zähigkeits- und Wärmeleitungsterme vernachlässigbar sind, und in einem inneren Bereich mit vernachlässigbaren mittleren Konvektionstermen. Die innere Schicht wird als eine Schicht mit konstantem Wärmestrom angenommen.

Eine Ähnlichkeitsanalyse ergibt allgemeingültige Geschwindigkeits- und Temperaturprofile in der äuß eren Schicht und in der Schicht mit konstantem Wärmestrom. Eine asymptotische Anpassung dieser Profile in einer Zwischenschicht (der Auftriebsunterschicht) liefert mit $H_\delta = g\beta F_0\delta^4/\alpha^3$ gegen ∞ analytische Ausdrücke für die Profile der Auftriebsunterschicht zu

$$\frac{T-T_w}{T_l} = K_2\left(\frac{y}{\eta}\right)^{-1/3} + A(Pr)$$

$$\frac{U}{U_l} = K_1\left(\frac{y}{\eta}\right)^{1/3} + B(Pr)$$

wobei K_1, K_2 allgemeine Konstanten und $A(Pr)$, $B(Pr)$ allgemeine Funktionen der Prandtl–Zahl sind. Asymptotische Wärmeübertragungs und Widerstandsgesetze ergeben sich zu

$$Nu_x = C'_H(Pr)H_x^{*1/4}, \quad \tau_w/\rho U_l^2 = C_f(Pr)$$

wobei $C'_H(Pr)$ auf einfache Weise mit $A(Pr)$ verknüpft ist. Schließlich wird gezeigt, daß wärmeleitende und thermisch-viskose Unterschichten, die durch eine lineare Geschwindigkeits- und Temperaturverteilung gekennzeichnet sind, an der Wand existieren. Alle Vorhersagen stimmen ausgezeichnet mit den zahlreichen experimentellen Daten überein.

ТЕОРИЯ СВОБОДНОКОНВЕКТИВНЫХ ТУРБУЛЕНТНЫХ ПОГРАНИЧНЫХ СЛОЁВ НА НАГРЕТЫХ ВЕРТИКАЛЬНЫХ ПОВЕРХНОСТЯХ

Аннотация — С помощью соображений размерности анализируется турбулентный естественно-конвективный пограничный слой на нагретой вертикальной поверхности. Показано, что полностью развитый турбулентный пограничный слой необходимо рассматривать состоящим из двух частей: внешней области, которая включает бóльшую часть пограничного слоя и в которой можно пренебречь вязкостью и теплопроводностью, и внутренней области, в которой можно пренебречь средними конвективными членами. Внутренняя область определяется как слой постоянного теплового потока.

С помощью анализа подобия получены универсальные профили скорости и температуры во внешней области слоя и в области постоянного теплового потока. Используя асимптотическое сращивание этих профилей в промежуточном слое (подслое) при $H_\delta \equiv g\beta F_0\delta^4/\alpha^3 \to \infty$, можно

получить следующие аналитические выражения для профилей в свободноконвективном подслое

$$\frac{T - T_w}{T_l} = K_2\left(\frac{y}{\eta}\right)^{-1/3} + A(Pr); \quad \frac{U}{U_l} = K_1\left(\frac{y}{\eta}\right)^{1/3} + B(Pr),$$

где K_1, K_2 — универсальные постоянные, а $A(Pr)$, $B(Pr)$ — универсальные функции критерия Прандтля. Асимптотические законы переноса тепла и трения записываются в виде $Nu_x = C'_H(Pr)H_x^{*1/4}$, $\tau_w/\rho U_l^2 = C_f(Pr)$, где $C'_H(Pr)$ связано простым соотношением с $A(Pr)$.

И наконец, показано, что на стенке имеют место теплопроводящий и термовязкий слои, характеризующиеся линейными законами изменения скорости и температуры. Полученные результаты расчётов хорошо согласуются с имеющимися в большом количестве экспериментальными данными.